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Folded localized excitations of the (2+1)-dimensional (M+N)-component AKNS system

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Abstract. Starting from the standard truncated Painlevé expansion and a multilinear variable separation approach, a quite general variable separation solution of the (2+1)-dimensional $(M+N)$-component AKNS (Ablowitz–Kaup–Newell–Segur) system is derived. In addition to the single-valued localized coherent soliton excitations like dromions, breathers, instantons, peakons, and a previously revealed chaotic localized solution, a new type of multi-valued (folded) localized excitation is obtained by introducing some appropriate lower-dimensional multiple valued functions. The folded phenomenon is quite universal in the real natural world and possesses quite rich structures and abundant interaction properties.

PACS. 05.45.Yv Solitons – 02.30.Jr Partial differential equations – 02.30.Ik Integrable systems

1 Introduction

In the study of nonlinear science, soliton theory plays a very important role and has been applied in almost all the natural sciences especially in all the physical branches such as fluid physics, condensed matter, biophysics, plasma physics, nonlinear optics, quantum field theory, particle physics, etc. [1]. Almost all the previous studies of soliton theory especially in higher dimensions are restricted to single-valued situations. However, real natural phenomena are very complicated. In various cases, it is even impossible to describe natural phenomena by single-valued functions. For instance, in the real natural world, there exist very complicated folded phenomena such as the folded protein [2], folded brain and skin surface, and many other kinds of folded biologic systems [3]. The simplest multi-valued (folded) waves may be the bubbles on (or under) a fluid surface. Various ocean waves are also folded waves.

To study the complicated folded natural phenomena is very difficult. At the present stage, it is impossible to give a complete view of the complicated folded natural phenomena. Similar to the single-valued case, the first important question we should and we can ask is: are there any stable multi-valued (folded) localized excitations? For convenience, we define the multi-valued localized excitations as folded solitary waves. Furthermore, if the interactions among the folded solitary waves are completely elastic, we call them foldons. As is known, the simplest foldons are the so-called loop solitons [4], which can be found in many (1+1)-dimensional integrable system [4] and have been applied in some possible physical fields like the string interaction with an external field [5], quantum field theory [6], and particle physics [7]. However, finding some folded localized excitations and/or foldons in higher-dimensional physical models is still open. Moreover, when saying that a model is integrable, one should emphasize two important facts. The first one is that we should point out in what special sense(s) is the model integrable. For instance, we say a model is Painlevé integrable if it possesses the Painlevé property and a model is Lax or IST (inverse scattering transformation) integrable if it has a Lax pair and it can be solved by the IST approach. A model integrable under some special cases may not be integrable under other cases. For instance, some Lax integrable models may not be Painlevé integrable [8]. The second fact is that for the general solution of a higher-dimensional integrable model, say, a Painlevé integrable model, there exist some characteristic, lower dimensional *arbitrary* functions. That means any lower dimensional multi-valued (folded) solutions can be used to construct exact solutions of higher-dimensional integrable models. In other words, any exotic behavior may propagate along the characteristics.

Motivated by these reasons, we take a general $(M+N)$-component (2+1)-dimensional AKNS system as a concrete example to study some types of (2+1)-dimensional folded

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localized excitations and/or foldons. To emphasize the importance of our present study, it is necessary to give the detailed background of the present physical model and review some previously obtained results, which are all single-valued localized situations. In (1+1)-dimensions, the AKNS system [9] is a most important physical model. The (1+1)-dimensional AKNS system had been extended in several different directions, say, the (1+1)-dimensional (1+1)-component AKNS system had been extended to the $(N+N)$-component (1+1)-dimensional AKNS system [10] and $(M+N)$-component (1+1)-dimensional AKNS system [11]. Several different types of (2+1)-dimensional integrable AKNS systems have also been obtained, say, the DS (Davey–Stewartson) type system [12] and the breaking soliton type system [13]. The (1+1)-dimensional AKNS system can be obtained from the usual symmetry constraint of the KP (Kadomtsev–Petviashvili) equation [10]. Recently, Lou and Hu have obtained a general $(M+N)$-component (2+1)-dimensional AKNS system,

$$ip_{it} + p_{ixx} + p_i u_x = 0, \quad i = 1, 2, ..., N, \tag{1a}$$

$$-iq_{jt} + q_{jxx} + q_j u_x = 0, \quad j = 1, 2, ..., M, \tag{1b}$$

$$u_y + \sum_{i=1}^{N} \sum_{j=1}^{M} a_{ij} p_i q_j = 0, \tag{1c}$$

from the inner parameter dependent symmetry constraints of the KP equation [11]. When we take $y = x$, $N = M = 1$, the system (1a) $\sim$ (1c) is reduced to the usual (1+1)-dimensional AKNS system. If q is selected as the complex conjugate and $M = N = 1$, then the system (1a) $\sim$ (1c) can be considered as the asymmetric part of the DS system. The so-called long-wave–short-wave interaction model is linked with equations (1a) $\sim$ (1c) by $N = M = 1$ by

$$p(x, y, t) = L(x, y + it, it) \equiv L(x', y', t'),$$
$$q(x, y, t) = S(x, y + it, it) \equiv S(x', y', t').$$

The Maccari system [14] is also a special case of systemé(1a) $\sim$ (1c) with $M = N = 2$ and $\{q_1 = p_1^*,\ q_2 = p_2^*\}$.

The main purpose of the present work is to find higher-dimensional multi-valued (folded) localized excitations for the significant (2+1)-dimensional $(M+N)$-component AKNS system. This paper is organized as follows. In Section 2, we apply the standard truncated Painlevé expansion and a multilinear variable separation approach (MLVSA) to solve the (2+1)-dimensional $(M+N)$-component AKNS system and obtain its exact excitation. In Section 3, we discuss some folded solitary waves (FSWs) and foldons based on the results obtained from the variable separation excitation for the (2+1)-dimensional $(M+N)$-component AKNS system. A brief summary and discussion is given in the last section.

2 The general variable separation solution of (2+1)-dimensional (M+N)-component AKNS system

To search for soliton excitations of a physical model, we can use different kinds of methods. One of the powerful methods is the multilinear variable separation approach (MLVSA), which was recently presented and successfully applied in some (2+1)-dimensional models [15–18]. Now we use this method to investigate the (2+1)-dimensional $(M+N)$-component AKNS system. To solve system (1a) $\sim$ (1c), we first substitute the following truncated Painlevé expansion into the original system

$$p_i = \frac{P_i}{f} + p_{i0}, \quad q_j = \frac{Q_j}{f} + q_{j0}, \quad u = \frac{2f_x}{f} + u_0 \tag{2}$$

from which the model yields the following bilinear form:

$$\left(iD_t + D_x^2 + u_{0x}\right) P_i \cdot f + p_{i0} D_x^2 f \cdot f = 0, \tag{3a}$$

$$\left(-iD_t + D_x^2 + u_{0x}\right) Q_j \cdot f + q_{j0} D_x^2 f \cdot f = 0, \tag{3b}$$

$$D_x D_y f \cdot f + \sum_{i=1}^{N} \sum_{j=1}^{M} a_{ij} \left(P_i Q_j + p_{i0} f Q_j + q_{j0} f P_i\right) = 0, \tag{3c}$$

where D_t, D_x and D_y are the standard Hirota's bilinear operator and $\{p_{i0}, q_{j0}, u_0\}$ are arbitrary seed solutions of the system (1a) $\sim$ (1c).

If we select the seed solution as

$$p_{i0} = q_{j0} = 0, \quad u_0 = u_0(x, t), \tag{4}$$

where $u_0 = u_0(x, t)$ is an arbitrary function of x and t, then the bilinear system (3a) $\sim$ (3c) can be solved by using the variable separation ansatz

$$f = a_1 F(x, t) + a_2 G(y, t) + a_3 F(x, t) G(y, t), \tag{5a}$$

$$P_i = F_{1i}(x, t) G_{1i}(y, t) \exp\left(iR_{1i}(x, t) + iS_{1i}(y, t)\right), \tag{5b}$$

$$Q_j = F_{2j}(x, t) G_{2j}(y, t) \exp\left(-iR_{2j}(x, t) - iS_{2j}(y, t)\right), \tag{5c}$$

where the space variables x and y have been totally separated into the functions $\{F, F_{1i}, F_{2j}, R_{1i}, R_{2j}\}$ and $\{G, G_{1i}, G_{2j}, S_{1i}, S_{2j}\}$, respectively.

Substituting the above ansatz into equations (3a) $\sim$ (3c) and using the fact that the space variables x and y have been separated into different functions $\{F, F_{1i}, F_{2j}, R_{1i}, R_{2j}\}$ and $\{G, G_{1i}, G_{2j}, S_{1i}, S_{2j}\}$, we can find that

$$G_{1i} = \frac{b_{1i}}{a_{1i}(t)}\sqrt{G_y}, \quad G_{2j} = \frac{b_{2j}}{a_{2j}(t)}\sqrt{G_y}, \tag{6a}$$

$$F_{1i} = a_{1i}(t)\sqrt{F_x}, \quad F_{2j} = a_{2j}(t)\sqrt{F_x}, \tag{6b}$$

$$S_{1i} = B(t) + s_{1i}(y), \quad S_{2j} = B(t) + s_{2j}(y), \tag{6c}$$

$$R_{1ix} = R_{2jx} \equiv R = -\frac{\left(a_2^2\alpha_0(t) + a_1a_2F_t + a_2\alpha_2(t)F + \alpha_1(t)F^2\right)}{2a_1a_2F_x}, \tag{6d}$$

$$G_t = \frac{G^2}{a_1^2}\left(a_3^3\alpha_0(t) - a_3\alpha_2(t) + \alpha_1(t)\right) + \frac{G}{a_1}\left(2a_3\alpha_0(t) - \alpha_2(t) + \alpha_0(t)\right), \tag{6e}$$

$$u_{0x} = \frac{1}{4a_1^2a_2^2F_x^2}\Big(a_1^2a_2^2F_t^2 + 2a_1a_2\left(a_2\alpha_0(t) + a_2\alpha_2(t)F + \alpha_1(t)F^2\right)F_t + a_1^2a_2^2\left(F_{xx}^2 - 2F_xF_{xxx} + 4\left(B_t + R_t\right)F_x^2\right) + a_2^2\alpha_2^2(t)F^2 + 2a_2\alpha_2(t)F\left(a_2^2\alpha_0(t) + F^2\alpha_1(t)\right) + \left(a_2^2\alpha_0(t) + \alpha_1(t)F^2\right)^2\Big), \tag{6f}$$

where b_{1i} and b_{2j} are arbitrary constants and $F(x,t)$, $a_{1i}(t)$, $a_{2j}(t)$, $s_{1i}(y)$, $s_{2j}(y)$, $B(t)$, $\alpha_0(t)$, $\alpha_1(t)$, $\alpha_2(t)$ are all arbitrary functions of the indicated variables with the condition

$$\sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}b_{1i}b_{2j}\exp\left(i\left(s_{1i}(y) - s_{2j}(y)\right)\right) = 2a_1a_2. \tag{7}$$

Hence, for the quantity $\nu \equiv \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}p_iq_j$, we have

$$\nu = \frac{2a_1a_2F_xG_y}{(a_1F + a_2G + a_3FG)^2}, \tag{8}$$

with F being an arbitrary function of x and t and $G = G(y,t)$ being an arbitrary solution of the Riccati equation (6e). After some slight modifications, one can find that the expression (8) is valid for many (2+1)-dimensional models like the DS equation, NNV system, ANNV equation, BK equation, and the higher-order BK equation, etc. [11,15–18]. We call all these models the MLVSA solvable models. Because some arbitrary characteristics, lower dimensional functions (like F), have been included in the "universal" formula (8), by selecting them appropriately, abundant stable localized structures have been revealed for these models. If we consider the boundary (or initial) condition of the given localized excitations, we can find that all the (2+1)-dimensional localized solutions of these models are caused by the suitable boundary (or initial) condition [19,20]. In other words, the richness of the localized excitations of the (2+1)-dimensional models results from the fact that arbitrary exotic behaviors can propagate along some special characteristics of the models. In a previous study [21], we had pointed out that some types of nonlocalized chaotic and periodic patterns may exist also in the (2+1)-dimensional soliton system because of some arbitrary characteristics functions that can be included in the special variable separation solutions. In the next section, we focus our attention on the possible multi-valued (folded) localized excitations, FSWs and foldons, constructed on the basis of the universal formula (8) and the interaction properties of these folded localized excitations for the (2+1)-dimensional $(M+N)$-component AKNS system.

3 Folded localized excitations for the (2+1)-dimensional (M+N)-component AKNS system

In order to construct some kinds of interesting folded localized excitations and/or foldons for the quantity ν, we first introduce some suitable multi-valued functions. For example,

$$G_y = \sum_{j=1}^{M} V_j(\eta + c_jt), \qquad y = \eta + \sum_{j=1}^{M} Y_j(\eta + c_jt), \tag{9}$$

where V_j and Y_j are localized excitations with the properties $V_j(\pm\infty) = 0, Y_j(\pm\infty) = \text{const}$. From equation (9), one can know that η may be a multi-valued function in some suitable regions of y by selecting the functions Y_j appropriately. Therefore, the function G_y, which is obviously an interaction solution of M localized excitations since the property $\eta\,|_{y\to\infty} \to \infty$, may be a multi-valued function of y in these areas though it is a single-valued function of η. Actually, most of the known multi-loop solutions are the special situation of equation (9). Similarly, we also treat the function $F(x,t)$ in this way:

$$F_x = \sum_{j=1}^{N} U_j(\xi + w_jt), \qquad x = \xi + \sum_{j=1}^{N} X_j(\xi + w_jt). \tag{10}$$

Now, if all the arbitrary functions in the universal formula (8) possess the forms similar to (9) with (10), then we can get various (2+1)-dimensional FSWs and/or foldons.

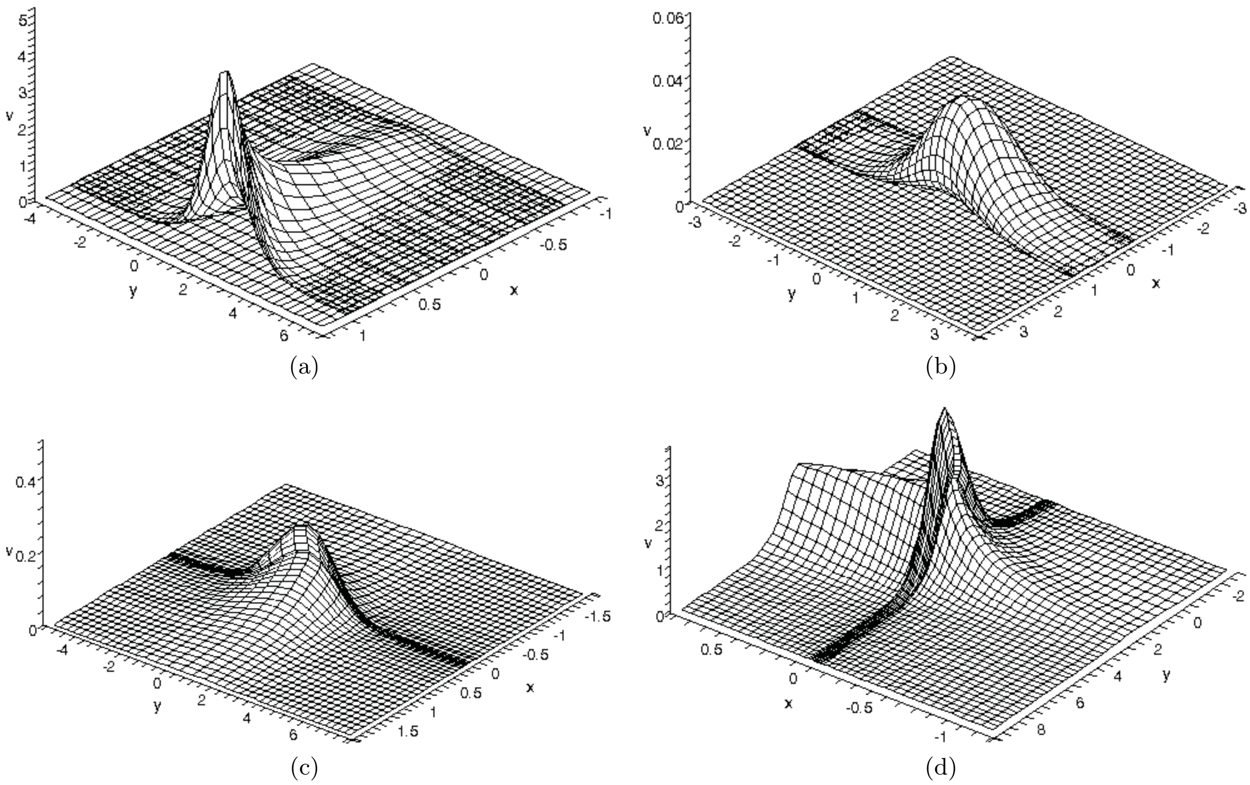


Fig. 1. Four typical folded solitary waves for the quantity ν determined by equation (8) at $t = 0$ together with (11–14) and $a_1 = a_2 = 1$, $a_3 = 1/25$ for (a) the "tent" shape, (b) the "worm" shape, (c) the "worm-dromion" shape, and (d) the "worm-solitoff" shape folded solitary wave, respectively.

In Figure 1, four typical folded solitary waves are plotted for the quantity ν determined by equation (8) with the function selections

$$\begin{aligned} F_x &= -\text{sech}^2(\xi + wt), \\ F &= \frac{2\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{5\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 1.9, \\ x &= \xi - 2.5\tanh(\xi + wt), \\ G_y &= -\text{sech}^2(\eta + ct), \quad G = -\frac{\sinh(\eta + ct)}{\cosh(\eta + ct)}, \quad y = \eta. \end{aligned} \tag{11}$$

$$\begin{aligned} F_x &= -\text{sech}^2(\xi + wt), \\ F &= \frac{2\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{5\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 8, \\ x &= \xi - 2.5\tanh(\xi + wt), \\ G_y &= -\text{sech}^2(\eta + ct), \quad G = -\frac{\sinh(\eta + ct)}{\cosh(\eta + ct)}, \quad y = \eta. \end{aligned} \tag{12}$$

$$\begin{aligned} F_x &= -10\text{sech}^2(\xi + wt), \\ F &= -\frac{7\sinh(\xi + wt)}{3\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{6\cosh^3(\xi + wt)} + 10, \\ x &= \xi - 1.15\tanh(\xi + wt), \\ G_y &= -\text{sech}^2(\eta + ct), \\ G &= -\frac{5\sinh(\eta + ct)}{3\cosh(\eta + ct)} - \frac{\sinh(\eta + ct)}{3\cosh^3(\eta + ct)}, \\ y &= \eta + \tanh(\eta + ct). \end{aligned} \tag{13}$$

$$\begin{aligned} F_x &= -\text{sech}^2(\xi + wt), \\ F &= -\frac{7\sinh(\xi + wt)}{30\cosh(\xi + wt)} + \frac{23\sinh(\xi + wt)}{60\cosh^3(\xi + wt)} + 2.03, \\ x &= \xi - 1.15\tanh(\xi + wt), \\ G_y &= -\text{sech}^2(\eta + ct), \\ G &= -\frac{5\sinh(\eta + ct)}{3\cosh(\eta + ct)} - \frac{\sinh(\eta + ct)}{3\cosh^3(\eta + ct)}, \\ y &= \eta + \tanh(\eta + ct). \end{aligned} \tag{14}$$

Figure 2 shows an other five typical folded solitary waves for the quantity ν determined by equation (8) with the function selections (15–19). However, the parameters

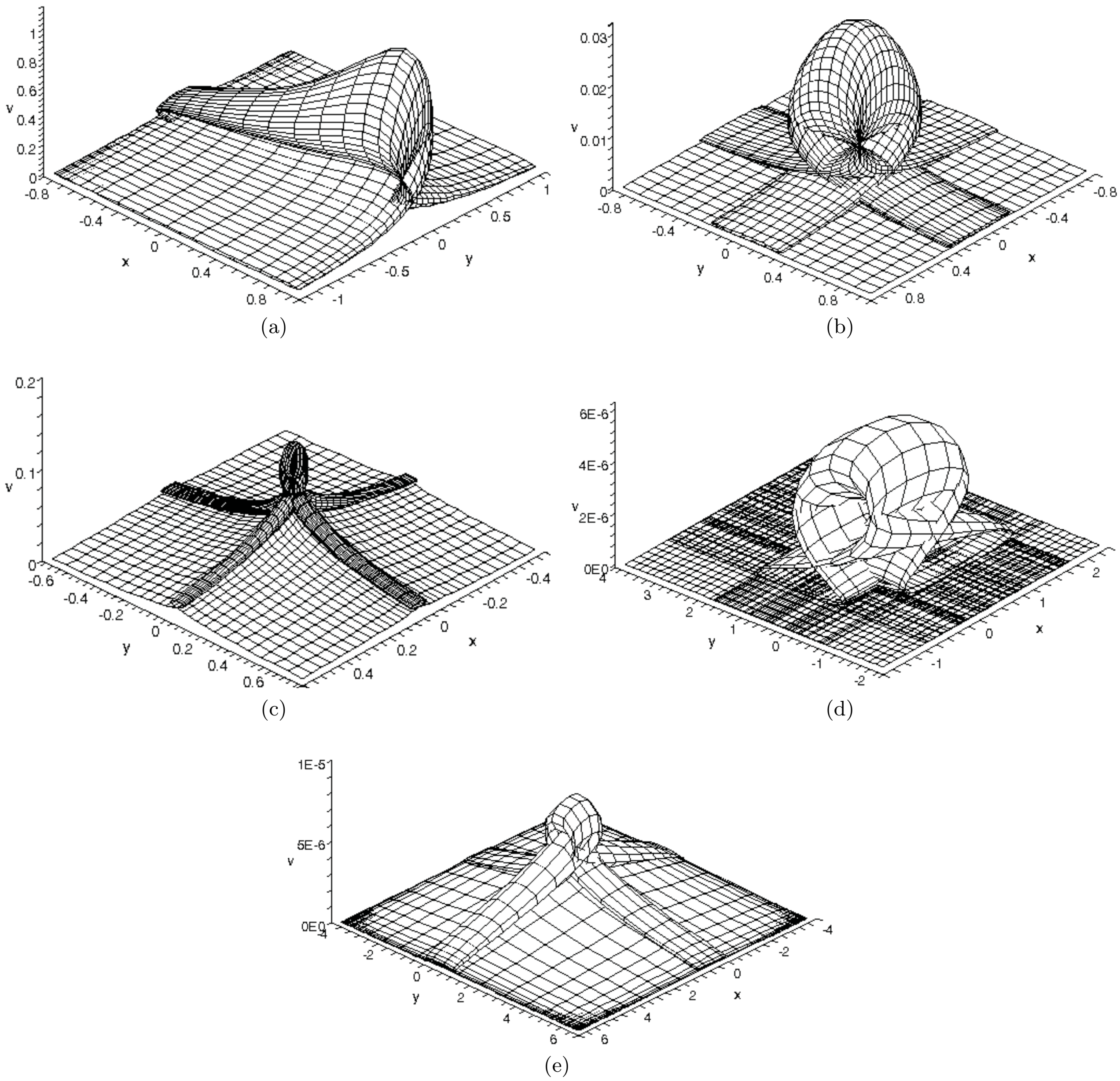


Fig. 2. Five typical folded solitary waves for the quantity ν determined by equation (8) at $t = 0$ with (15–19) and $a_1 = a_2 = 1$, $a_3 = 1/25$ are shown in (a), (b), (c), (d), and (e), respectively.

are chosen such that both F and G are multi-valued

$$F_x = -\text{sech}^2(\xi + wt),$$

$$F = -\frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{7\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 1.9,$$

$$x = \xi - 1.4\tanh(\xi + wt),$$

$$G_y = -\text{sech}^2(\eta + ct),$$

$$G = \frac{2\sinh(\eta + ct)}{3\cosh(\eta + ct)} + \frac{5\sinh(\eta + ct)}{6\cosh^3(\eta + ct)},$$

$$y = \eta - 2.5\tanh(\eta + ct). \qquad (15)$$

$$F_x = -\text{sech}^2(\xi + wt),$$

$$F = \frac{\sinh(\xi + wt)}{15\cosh(\xi + wt)} + \frac{8\sinh(\xi + wt)}{15\cosh^3(\xi + wt)} + 8,$$

$$x = \xi - 1.6\tanh(\xi + wt),$$

$$G_y = -\text{sech}^2(\eta + ct),$$

$$G = \frac{\sinh(\eta + ct)}{15\cosh(\eta + ct)} + \frac{8\sinh(\eta + ct)}{15\cosh^3(\eta + ct)},$$

$$y = \eta - 1.6\tanh(\eta + ct). \qquad (16)$$

$$F_x = -\mathrm{sech}^2(\xi + wt),$$

$$F = -\frac{7\sinh(\xi+wt)}{30\cosh(\xi+wt)} + \frac{23\sinh(\xi+wt)}{60\cosh^3(\xi+wt)} + 4,$$

$$x = \xi - 1.15\tanh(\xi+wt),$$

$$G_y = -\mathrm{sech}^2(\eta + ct),$$

$$G = -\frac{7\sinh(\eta+ct)}{30\cosh(\eta+ct)} + \frac{23\sinh(\eta+ct)}{60\cosh^3(\eta+ct)},$$

$$y = \eta - 1.15\tanh(\eta+ct). \quad (17)$$

$$F_x = \mathrm{sech}^2(\xi + wt),$$

$$x = \xi + 2\tanh(\xi+wt) + \tanh^2(\xi+wt) - 5.5\tanh^3(\xi+wt),$$

$$G_y = \mathrm{sech}^2(\eta+ct) + \mathrm{sech}^6(\eta+ct),$$

$$y = \eta + 2\tanh(\eta+ct) + \tanh^2(\eta+ct) - 5.5\tanh^3(\eta+ct). \quad (18)$$

$$F_x = \mathrm{sech}^2(\xi + wt),$$

$$x = \xi + 2\tanh(\xi+wt) + \tanh^2(\xi+wt) - 10\tanh^3(\xi+wt),$$

$$G_y = \mathrm{sech}^2(\eta+ct) + \mathrm{sech}^6(\eta+ct),$$

$$y = \eta + 2\tanh(\eta+ct) + \tanh^2(\eta+ct) - 10\tanh^3(\eta+ct). \quad (19)$$

4 Interaction properties of (2+1)-dimensional localized excitations

Fortunately, owing to the arbitrary value of the function in the expression (8), we have constructed quite rich folded solitary waves. Now, one of the most important problems which should be discussed is whether these types of localized excitations are solitons. Particularly, are these FSWs foldons? To find the answer, we have to study the interaction properties among these types of localized excitations, then some concrete interaction example of FSWs and foldons are given. In principle, following the general ideas introduced in reference [19], one could investigate the stability properties of the solutions presented in this paper and their relevance as asymptotic states for suitable initial boundary value problems. However, here, we study only the interaction behavior among the localized solutions by studying the asymptotic property of the universal formula (8) because these formulas are valid for more than one system.

4.1 Asymptotic behaviors of the localized excitations produced from (8)

In general, if the function F and G are selected as localized solitonic excitations with

$$F\Big|_{t\to\mp\infty} = \sum_{i=1}^{M} F_i^{\mp}, \quad F_i^{\mp} \equiv F_i\left(x - c_i t + \delta_i^{\mp}\right), \quad (20)$$

$$G\Big|_{t\to\mp\infty} = \sum_{j=1}^{N} G_j^{\mp}, \quad G_j^{\mp} \equiv G_j\left(y - C_j t + \Delta_j^{\mp}\right), \quad (21)$$

where $\{F_i, G_j\}\,\forall i$ and j are localized functions, then the physical quantity ν expressed by equation (8) delivers $M \times N$ (2+1)-dimensional localized excitations with the asymptotic behaviour

$$\nu\,|_{t\to\mp\infty} \to \sum_{i=1}^{M}\sum_{j=1}^{N}\left\{\frac{2a_1a_2F_{ix}^{\mp}G_{jy}^{\mp}}{(a_1(F_i^{\mp}+f_i^{\mp})+a_2(G_j^{\mp}+g_j^{\mp})+a_3(F_i^{\mp}+f_i^{\mp})(G_j^{\mp}+g_j^{\mp}))^2}\right\}$$
$$\equiv \sum_{i=1}^{M}\sum_{j=1}^{N}\nu_{ij}^{\mp}\left(x - c_i t + \delta_i^{\mp}, y - C_j t + \Delta_j^{\mp}\right) \equiv \sum_{i=1}^{M}\sum_{j=1}^{N}\nu_{ij}^{\mp}, \quad (22)$$

where

$$f_i^{\mp} = \sum_{j<i} F_j\,(\mp\infty) + \sum_{j>i} F_j\,(\pm\infty), \quad (23)$$

$$g_i^{\mp} = \sum_{j<i} G_j\,(\mp\infty) + \sum_{j>i} G_j\,(\pm\infty), \quad (24)$$

and we have assumed without loss of generality, $C_i > C_j$ and $c_i > c_j$ if $i > j$.

It can be deduced from expression (22) that the ijth localized excitation ν_{ij} preserves its shape during the interaction if

$$f_i^+ = f_i^-, \quad (25)$$

$$g_j^+ = g_j^-. \quad (26)$$

Meanwhile, the phase shift of the ijth localized excitation ν_{ij} reads

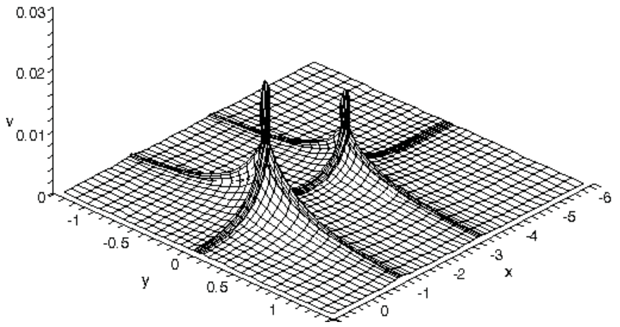
$$\delta_i^+ - \delta_i^- \quad (27)$$

in the x direction and

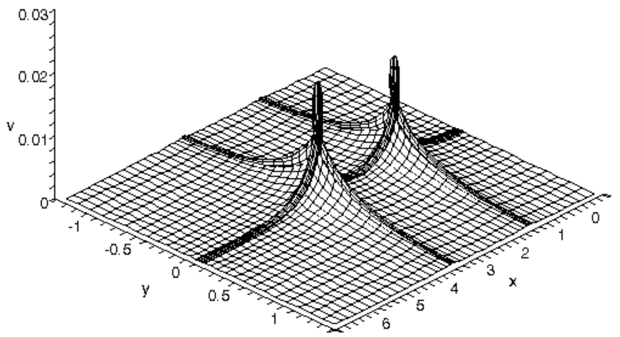
$$\Delta_j^+ - \Delta_j^- \quad (28)$$

in the y direction.

The above discussions demonstrate that localized solitonic excitations for the universal quantity ν can be constructed without difficulty via the (1+1)-dimensional localized excitations with the properties (20), (21), (25), and (26). As a matter of fact, any localized solutions (or their derivatives) with completely elastic (or not completely elastic or completely inelastic) interaction behaviors of any known (1+1)-dimensional integrable models



(a)



(b)

Fig. 3. Pre- and post-interaction of two folded solitary waves at time (a) $t=-4.5$, and (b) $t=4.5$ for the quantity ν determined by equation (8) with the selections (29) and $a_1=a_2=1$, $a_3=1/25$.

can be utilized to construct (2+1)-dimensional localized solitonic solutions with completely elastic ($f_i^+=f_i^-$, $g_j^+=g_j^-$ for all i,j) (or not completely elastic or completely inelastic ($f_i^+\neq f_i^-$, $g_j^+\neq g_j^-$ at least for one of i,j)) interaction properties. In order to see the interaction behaviors among multi-valued (folded) localized excitations more directly and visually, we investigate some special examples by fixing the arbitrary functions F and G in equation (8).

4.2 Example of the interaction between two FSWs for (2+1)-dimensional (M+N)-component AKNS system

Some examples of the single FSW have been discussed in the last section and the general aspect of the related multiple FSWs and foldons has been given in Section 4.1. Here we write and plot two more special two-FSW (Sect. 4.2) and two-foldon (Sect.4.3) solutions for the universal quantity ν.

Figure 3 is a pre- and post-interaction plot of the two folded solitary waves for the quantity ν determined by equation (8) with the selections

$$F_x=-12\mathrm{sech}^2(\xi)-10\mathrm{sech}^2(\xi-t),$$

$$x=\xi-1.15\tanh(\xi)-1.15\tanh(\xi-t),$$

$$G_y=-\mathrm{sech}^2(\eta),\quad y=\eta-1.15\tanh(\eta). \tag{29}$$

From Figures 3a and 3b, we know that the quantity ν (8) with (29) expresses a special two-FSW solution in that the interaction between them is inelastic. Actually, the completely elastic interaction condition (25) is not satisfied for the solution (8) with (29).

4.3 Example of the interaction between two foldons for (2+1)-dimensional (M+N)-component AKNS system

According to the general discussions in Section 4.1, in order to find foldons, the functions F and G must be selected in such a way that the conditions (25–26) are satisfied.

Figure 4 shows evolution plots of two foldons for the quantity ν determined by equation (8) with the selections

$$F_x=-\frac{4}{5}\mathrm{sech}^2(\xi)-\frac{1}{2}\mathrm{sech}^2(\xi-t),$$

$$x=\xi-1.5\tanh(\xi)-1.5\tanh(\xi-t),$$

$$G_y=-\mathrm{sech}^2(\eta),\quad y=\eta-2\tanh(\eta). \tag{30}$$

Since the completely elastic interaction condition (25–26) is really satisfied for both the static excitation and the moving one, the solution (8) with (30) is a genuine two-foldon solution.

5 Summary

In summary, with the help of the standard truncated Painlevé expansion and a multilinear variable separation approach, the (2+1)-dimensional $(M+N)$-component AKNS system is solved. Abundant localized coherent soliton structures of the solution (8) like dromions, lumps, ring solitons, breathers, instantons, solitoffs, fractal and chaotic localized excitations, etc., can be easily constructed by selecting arbitrary functions appropriately. Except for the single-valued localized excitations, we find a new type of multi-valued localized excitations, i.e. folded solitary wave and/or foldon excitations for the (2+1)-dimensional $(M+N)$-component AKNS system. To our knowledge, the folded solitary wave and/or foldon excitations for the (2+1)-dimensional $(M+N)$-component AKNS system have not reported previously in the literatures.

On the one hand, there are a large number of complicated "folded" and multi-valued phenomena in the real natural world. On the other hand, there is no good analytical way to treat these kinds of complicated phenomena.

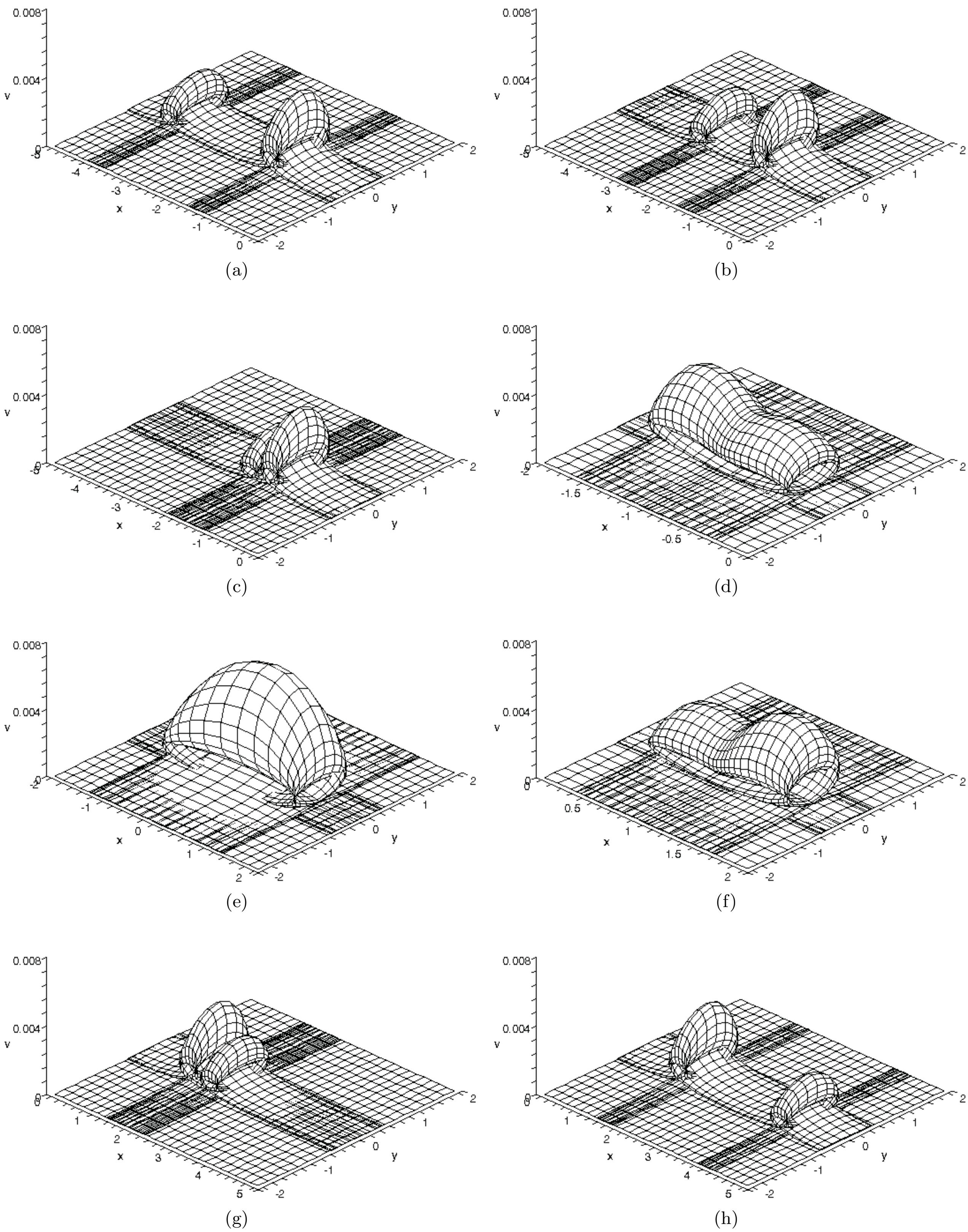


Fig. 4. Evolution plots of two foldons for the quantity ν determined by equation (8) with the selections (30) and $a_1 = a_2 = 1$, $a_3 = 1/25$ at time (a) $t = -5.5$, (b) $t = -4.5$, (c) $t = -3.5$, (d) $t = -2$, (e) $t = 0$, (f) $t = 2$, (g) $t = 3.5$, and (h) $t = 5.5$.

This work is only a first attempt to find some types of possible stable multi-valued localized excitations, folded solitary waves and foldons, for some real physical models. Further study to find the localized excitations such as new types of folded solitary waves, foldons and their applications in reality is still necessary.

In references [19] and [22], the author pointed out that the localized solutions of the DS equation, say, dromions, can be remote controlled by choosing a suitable motion of the boundaries. In reference [15], Tang and Lou also pointed out that though the localized excitations such as the dromions, lumps, ring solitons, peakons and foldons proposed here possess zero boundary conditions for the quantity ν, the boundary conditions for other quantities, say, the mean flow for the DS model, are not identically zero. The different selections of the arbitrary functions F and G in (8) correspond to the different selections of the boundary conditions of those fields (or potentials) with nonzero boundary conditions and vice versa. That means, in some sense, the dromions, foldons, and other types of localized excitations for some physical quantities are remote controlled by some other quantities (or potentials). This fact hints that it is possible for one to observe the dromions, foldons, and other types of localized excitations from the systems governed by the MLVSA solvable models by inputting suitable boundary conditions. For foldons, the input boundaries may be selected as (1+1)-dimensional loop solitons.

Since the excitation (8) is a "universal" formula for many (2+1)-dimensional physical models which are widely applied in many physical fields, we do believe that foldons are useful in the studies on the complicated "folded" natural world. Both the "universal" formula and the general (or special) foldons especially their possible real applications should be studied further.

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